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Periodicity of Y-Systems and Flat Connections

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Abstract. We give a proof of the periodicity of Zamolodchikov's Y-system in the AxA case using an interpretation of the system as a condition of flatness of a certain graph connection. In our approach, the periodicity property appears as an identity among representations of a matrix as products of two-diagonal matrices.

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1. Introduction

The Y-system is a multi-dimensional rational recursion, which first appeared in theoretical physics. It was introduced by Zamolodchikov [14], who was motivated by ideas of deformed conformal field theories (see also [10] for connections with integrable systems). The recursion was generalized by Ravanini et al. [11], and in this general form it is parameterized by a pair of Dynkin diagrams. In the present paper we restrict our attention to the case of Dynkin diagrams of type A. The recursion starts with a certain number of free parameters, and progresses along according to a set of relations (see (1)). The conjecture is that the recursion eventually returns to its original starting conditions (Theorem 3.1).

To prove this periodicity property turned out to be remarkably difficult, in part, because it was not clear what mathematical tools one could employ. In the case $A_1 \times A_k$ the conjecture was proved by Frenkel and Szenes [5] and Gliozzi and Tateo [6], using a rational parameterization based on continued fractions and hyperbolic geometry, respectively.

The cases $A_1 \times \Delta$, where Δ is any Dynkin diagram was treated by Fomin and Zelevinsky in [3], where they successfully related the system to their theory of cluster algebras. This case provided one of the basic examples of the theory. In a remarkable recent paper [4], Fomin and Zelevinsky also show that if one replaces

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the exponents in the recursion of the $A_1 \times M$ case with a matrix M in a rather wide class, then the system will be periodic only if M comes from a Dynkin diagram.

We also note that the Y-system of Zamolodchikov has been linked to identities of the dilogarithm functions [2,5,6].

The general case of the product of two Dynkin diagrams has been open for more than 10 years now, even in the case $A_k \times A_r$. In this case the Y-system is not clearly related to the Fomin–Zelevinsky theory cluster algebras.

In this paper, we give a proof of the periodicity of the $A_k \times A_r$ system, using an interpretation of the system as a system of flat connections on a graph. This idea goes back to the papers of Tarasov [12] and Bazhanov and Kashaev [1] (cf. also [8]). In the context of Y-systems, the zero-curvature condition appears in the work of Krichever et al. [9], where the connection of this system to other important equations of the theory of integrable systems, such as the Hirota's bilinear difference equations (HBDE) is also explored. Using these results, one can easily translate the periodicity of the Y-system to the periodicity of HBDE and the periodicity of other families of related systems of equations. The determinantal formulas of [9] for the solutions of the infinite Y-system could serve as a starting point to an approach for proving periodicity in the case of general Dynkin diagrams.

When this work was substantially completed, we learned of another proof of the periodicity of Y-systems for the $A \times A$ case by Volkov [13]. Volkov's proof is rather different; it uses an explicit parameterization. The two proofs are of independent interest. It would be interesting to see if either of these proofs could be generalized to the arbitrary Dynkin diagram case.

We should also mention that a similar periodicity phenomenon has been observed by Henriques [7].

2. The Infinite System

Consider the 3-dimensional lattice $\Lambda = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. We will need to visualize this lattice, and our convention will be that the first coordinate is horizontal (West-to-East), the second is vertical (North-to-South), and the third is up-down (plus-minus). The infinite Y-system is an algebraic system with infinitely many variables and infinitely many relations, both indexed by the lattice points in Λ.

It is best to think of the variables as formal ones, but for simplicity of notation we will consider them as having actual complex values. Thus consider the space of complex valued functions on Λ

$$F(\Lambda) = \{Y : \Lambda \to \mathbb{C}^*\}.$$

Define the shift operators on this space: for $\mathbf{m} = (n, i, j) \in \Lambda$, let

$$Y^W(\mathbf{m}) = Y(n-1, i, j), \quad Y^N(\mathbf{m}) = Y(n, i-1, j), \quad Y^E(\mathbf{m}) = Y(n+1, i, j),$$
$$Y^S(\mathbf{m}) = Y(n, i+1, j), \quad Y_+(\mathbf{m}) = Y(n, i, j+1), \quad Y_-(\mathbf{m}) = Y(n, i, j-1).$$

Introduce the so-called Y-*system*:

$$Y^W Y^E = \frac{(1+Y^N)(1+Y^S)}{(1+1/Y_+)(1+1/Y_-)}. \tag{1}$$

Thus indeed, for each site $\mathbf{m} \in \Lambda$, there is one value, $Y(\mathbf{m})$, and also one relation, $\mathrm{Rel}(\mathbf{m})$, relating the values of Y on the six neighboring sites.

Three quick observations about this system.

Decoupling The lattice Λ is the disjoint union of the lattices

$$\Lambda^{\text{even}} = \{(n,i,j);\ i+j+n \text{ even}\} \quad \text{and} \quad \Lambda^{\text{odd}} = \{(n,i,j);\ i+j+n \text{ odd}\}.$$

The system clearly decouples into two, one with variables parameterized by Λ^{even}:

$$F^{\text{even}} = \{Y : \Lambda^{\text{even}} \to \mathbb{C}^*\},$$

and relations parameterized by Λ^{odd}, and another, equivalent system with the roles of *even* and *odd* exchanged. From now on we will only consider the even system.

Rationality Again, by inspecting the relations (1), one can easily see that the variables $\{Y(0,i,j), Y(1,i,j);\ i,j \in \mathbb{Z}\}$ are independent and determine the rest of the variables uniquely. More precisely, using (1), any variable of the system may be expressed as a rational function of the variables with $n=0,1$. Moreover, a generic function $\hat{Y} : \{(n,i,j);\ n=0 \text{ or } 1\} \to \mathbb{C}^*$ extends to a unique element of $F(\Lambda)$ satisfying (1).

Symmetry By replacing Y with $1/Y$ the system turns into another one, which has exactly the same form, with the North-South direction exchanged with the up-down direction.

3. The Truncation and the Formulation of the Conjecture

Now we introduce a truncated version of (1). Let

$$\Lambda_{rk} = \{(n,i,j);\ 1 \le i \le r,\ 1 \le j \le k\},$$

be the truncated lattice and denote by $F(\Lambda_{rk})$ the corresponding function space. For each $\mathbf{m} \in \Lambda_{rk}$ we impose the relation $\mathrm{Rel}(\mathbf{m})$ given in (1), with the *convention* that the factors with $i, j < 1$, $i > r$ and $j > k$, are simply omitted from the equations. Another way to put this is to set the boundary conditions

$$Y(n,0,j) = Y(n,r+1,j) = 1/Y(n,i,0) = 1/Y(n,i,k+1) = 0.$$

Just as in the infinite case, it is not hard to see that this system still has the decoupling, rationality and symmetry properties, where this latter one now exchanges r and k. In particular, we will only consider the even system $F(\Lambda_{rk}^{\text{even}})$, whose relations are parameterized by $\Lambda_{rk}^{\text{odd}}$, and without loss of generality we can assume that $r \geq k$.

Now we are ready to formulate the periodicity conjecture. Define a twisted shift in the truncated lattice Λ_{rk} via the formula

$$\sigma : (n, i, j) \mapsto (n+r+k+2, r+1-i, k+1-j).$$

This is thus a combination of two central symmetries of the Dynkin diagrams A_r and A_k, and a translation in the $\mathbb{Z}$-direction by $r+k+2$.

THEOREM 3.1. *Consider a solution of the truncated* Y-*system, which is a map* $Y : \Lambda_{rk} \to \mathbb{C}^*$ *satisfying the Equations* (1) *with the convention described above. Then we have* $Y^\sigma = Y$, *i.e.*

$$Y(n, i, j) = Y(n+r+k+2, r+1-i, k+1-j).$$

Note that σ^2 is simply translation by $2(r+k+2)$, hence the system is indeed periodic. Recall that according to the rationality property above, starting from a generic set of values of $[Y(0, i, j), Y(1, i, j)]$ we obtain a solution of the truncated Y-system, but it is not at all clear why such system needs to be periodic.

4. Flat Connections

4.1. THE SHIFTED SYSTEM

It is convenient to introduce the following shifted form of our system. Let $z_j(n, i) = -Y(n, i-j, j)$. Our system (1) now looks as follows:

$$z^W z^E = \frac{1-z_-^N}{1-1/z^N}\,\frac{1-z_+^S}{1-1/z^S}. \tag{2}$$

In this paragraph, we, carefully and mechanically, rewrite all the statements and observations of the previous section in this shifted form. While this may seem tedious, without these formulas the proof will be difficult to follow.

To define the property of **decoupling**, consider the lattices

$$\hat\Lambda^{\text{even}} = \{(n, i, j);\ i+n \text{ even}\} \quad \text{and} \quad \hat\Lambda^{\text{odd}} = \{(n, i, j);\ i+n \text{ odd}\}.$$

Then again the system decouples into two, with the variables sitting on $\hat\Lambda^{\text{even}}$ related by equations indexed by $\hat\Lambda^{\text{odd}}$.

The truncated configurations look more complicated. We have the truncated lattices

$$\hat\Lambda_{rk}^{\text{even}} = \{(n, i, j) \in \hat\Lambda^{\text{even}};\ 1 \leq j \leq k,\ j+1 \leq i \leq j+r\}$$

and

$$\hat{\Lambda}_{rk}^{\mathrm{odd}} = \{(n,i,j) \in \hat{\Lambda}^{\mathrm{odd}};\ 1 \le j \le k,\ j+1 \le i \le j+r\}.$$

The index j thus takes the values $j=1,\dots,k$, and for $j=1$ we have

$$z^W z^E = \frac{1}{1-1/z^N} \frac{1-z_+^S}{1-1/z^S}, \tag{3}$$

while for $j=k$,

$$z^W z^E = \frac{1-z_-^N}{1-1/z^N} \frac{1}{1-1/z^S}. \tag{4}$$

The situation for the index i is a bit different: it takes the values $i=j+1,\dots,j+r$, and for $i=j+1$ we have

$$z^W z^E = (1-z_-^N) \frac{1-z_+^S}{1-1/z^S}, \tag{5}$$

while for $i=j+r$

$$z^W z^E = \frac{1-z_-^N}{1-1/z^N} (1-z_+^S), \tag{6}$$

The four additional cases for the four edges are similar, with two factors omitted. For example, for $j=1$, $i=j+1$, we have

$$z^W z^E = \frac{1-z_+^S}{1-1/z^S}. \tag{7}$$

4.2. THE Γ- SYSTEM

An important step in our proof is the reinterpretation of the z-system as the conditions of flatness of a connection defined on a certain graph (cf. Sect. 1 for references). Again, first we consider the simpler infinite case.

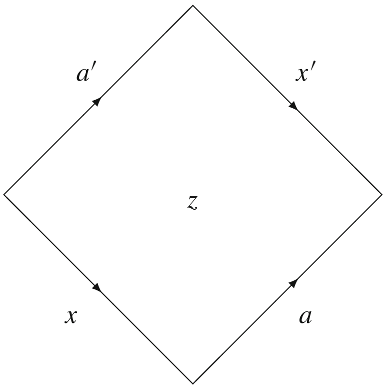
We introduce another infinite system now, which we will call the Γ-system. Formally, the variables of this system live on the points of the shifted lattice $v+\Lambda$, where $v=(-1/2,-1/2,0)$. We will use different symbols and conventions for the variables corresponding to $v+\hat{\Lambda}^{\mathrm{even}}$ and $v+\hat{\Lambda}^{\mathrm{odd}}$. Assuming that $n+i$ is even, we will write

$a_j(n,i)$ for the variable at $(n+1/2, i+1/2, j) \in v+\hat{\Lambda}^{\mathrm{even}}$

and

$x_j(n,i)$ for the variable at $(n-1/2, i+1/2, j) \in v+\hat{\Lambda}^{\mathrm{odd}}$.

We can also think of each variable of this new system as sitting on an interval joining two points of $\hat{\Lambda}^{\mathrm{odd}}$; in this picture then, the points of $\hat{\Lambda}^{\mathrm{even}}$ are the centers of squares formed by these intervals.



Now consider the directed graph Γ obtained as the projection of these intervals onto the ni-plane, with directions chosen eastward: northeast for the a-edges, and southeast for the x-edges. Associate to each SE edge the infinite matrix $X(i, j)$ which has 1s on the diagonal and $x_j(n, i)$ as the $(j, j+1)$ entry. Also, associate to a NE edge the matrix $A(n, i)$, which has $a_j(n, i)$ as the jth diagonal entry and all the entries under the diagonal are 1s. The rest of the entries vanish. These matrices define a connection on the graph Γ with values in $GL(\infty)$, and the *equations of the* Γ-*system* say that this connection is flat, i.e.

$$XA = A'X', \tag{8}$$

where we used the notation $A' = A^{NW}$ and $X' = X^{NE}$.

Explicitly, in terms of the entries, the equations look like

$$x + a = a' + x'_-, \quad xa_+ = a'x'. \tag{9}$$

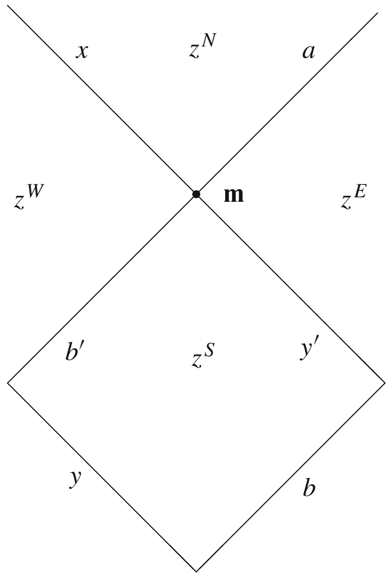
Here we think of x as a function on $\hat{\Lambda}^{\text{even}}$, a is a function on $\hat{\Lambda}^{\text{odd}}$, and again, $a' = a^{NW}$ and $x' = x^{NE}$.

Again, we have the rationality property of this system: now this means that the values of $x(0, i, j)$ and $a(0, i, j)$ are independent, and that all other values of x and a are rational functions of these values.

PROPOSITION 4.1. *Any* Γ-*system gives rise to a* z-*system via the formula*

$$z = \frac{x}{a'}, \tag{10}$$

and all z-*systems arise this way.*



Proof. Let (x, a) be a solution of the Γ-system, and let $\mathbf{m}$ be a lattice point in $\hat{\Lambda}^{\mathrm{odd}}$ marking one of the relations in (2). To simplify our notation, we will denote the variables surrounding z^N by x, a, x', a' as usual, while for the variables surrounding z_S we will use y, b, y', b'.

Now we compute the factors in (2). We have

$$\frac{1-z_+^S}{1-1/z^S} = \frac{1-y_+/b'_+}{1-b'/y} = \frac{b'_+ - y_+}{y-b'} \cdot \frac{y}{b'_+} = -\frac{y}{b'_+},$$

where we used the first equation in (9). For the other case, observe that the multiplicative equation in (9) maybe rewritten as $x/a' = x'/a_+$. Then

$$\frac{1-z_-^N}{1-1/z^N} = \frac{1-x'_-/a}{1-a'/x} = \frac{a-x'_-}{x-a'} \cdot \frac{x}{a} = -\frac{x}{a}. \tag{11}$$

Finally, note that

$$z^W = \frac{x}{b'_+}, \quad \text{and} \quad z^E = \frac{y'}{a}.$$

Now the Equation (2) immediately follows. □

Thus we have managed to interpret the Y-system as a system flat graph connections. Our next goal is to impose boundary conditions on the Γ-system in such a way that they induce the truncation of the z-system.

Imposing the condition in the j-direction is very natural: consider the Γ-system (8), with $k+1$-by-$k+1$ matrices instead of infinite ones. Thus now we have

$$X(n,i) = \begin{pmatrix} 1 & x_1 & 0 & \dots & \dots & 0 \\ 0 & 1 & x_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & x_{k-1} & 0 \\ 0 & \dots & \dots & 0 & 1 & x_k \\ 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix}$$

and

$$A(n,i) = \begin{pmatrix} a_1 & 0 & 0 & \dots & \dots & \dots & 0 \\ 1 & a_2 & 0 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & a_{k-1} & 0 & 0 \\ 0 & \dots & \dots & 0 & 1 & a_k & 0 \\ 0 & \dots & \dots & \dots & 0 & 1 & a_{k+1} \end{pmatrix}.$$

For this truncated system the index j for the a variable runs from $j=1$ to $k+1$, while for the variable x this interval is $j=1,\dots,k$.

This truncation of the Γ-system means that for $j=1$ we have $x+a=a'$, while for $j=k+1$ we have $a=a'+x'_{-}$. Now it is easy to check that the two equations (3) and (4) follow from this truncated Γ-system exactly as in the proof of Proposition 4.1

Now we come to the truncation in the i direction. Here the situation will be a not as pleasant, because we will not be able to interpret the resulting system as a flat connection on a graph immediately.

The truncated Γ-system is defined as follows:

$$x_j(n,i) \text{ is defined for } 1 \le j \le k,\ j \le i \le j+r,$$

while

$$a_j(n,i) \text{ is defined for } \begin{cases} j=1, & 1 \le i \le r+1, \\ 1 \le j \le k, & j \le i \le j+r-1, \\ j=k+1, & k \le i \le k+r. \end{cases}$$

The relations (9) are truncated as follows:

$$\begin{cases} x+a=a' & \text{if } j=1, \\ a=a'+x'_{-} & \text{if } j=k+1, \\ x+a=x'_{-} & \text{if } i=j, \\ x=a'+x'_{-} & \text{if } i=j+r. \end{cases} \tag{12}$$

Now we claim

PROPOSITION 4.2. *Consider the truncated* Γ-*system described above. Then the system* $z=x/a'$ *satisfies the equations of the truncated* z-*system. The map from truncated* Γ-*systems to the truncated* z-*systems is surjective.*

The proof is identical to that of Proposition 4.1. We would like to note that the variables $x_j(0,i)$, $a_j(0,i)$ are still independent, but now they do not necessarily determine the rest of the variables. Sometimes one has the freedom of choosing some of the edge a variables as the recursion progresses, but this fact does not influence our result.

5. Boundary Conditions and Permutations

As we mentioned above, the problem is that these truncated Γ-systems are cannot be interpreted as flat connections since for certain values of i not all entries of the corresponding matrices are defined.

The key idea is that our boundary conditions at these partially defined matrices force a certain matrix transformation which then quickly leads to the proof of the periodicity of the system.

We demonstrate this transformation for the $k=2$ case first. The general case is analogous. It is described in Proposition 5.1.

Thus we have a truncated system with $k=2$, i.e. with 3-by-3 matrices, and consider the $i=1$ case, which is the critical one, because the system is not defined for $j=2$. In other words, fixing $i=2$, we have x_1, x_2, a_1, a_2, a_3 and a'_1, x'_1 but not a'_2, x'_2, a'_3. Now we are unable to write down our usual matrix equation $XA=A'X'$. However, taking into account the truncated Γ-equations (12), concretely

$$x_1+a_1=a'_1, \quad x_2+a_2=x'_1, \quad x_1a_2=a'_1x'_1,$$

we have

$$XA=\begin{pmatrix} a'_1 & 0 & 0 \\ 1 & x_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & a_3 \\ 0 & 1 & a_3 \end{pmatrix}\begin{pmatrix} 1 & x'_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This equality can be checked by multiplying the matrices and comparing the corresponding entries. Clearly, this equality is local, i.e. we can write it down for any k. The structure is described in the following statement.

PROPOSITION 5.1. *For* $i=k$ *we will have*

$$XA=A'_{\oplus 1}\widehat{\Sigma}_{k,k+1}X'_{\oplus 1} \tag{13}$$

where A' *and* X' *are* k-*by*-k *matrices, whose entries* (*apart from the lower-left entry of* A') *are exactly those coming from the* $i=k-1$ *part of the system, and the matrix*

$\widehat{\Sigma}_{k,k+1}$ is identity on the first $k-1$ coordinates and a 2-by-2 transformation in the last two coordinates with the (k,k) entry vanishing.

Here we used the notation $M_{\oplus d}$ taking the direct sum of a matrix M with the identity matrix in dimension d. When $d=1$, this means adding a row and a column of zeros to M and then changing the lower-right entry to 1.

By iterating this transformation, we arrive at the following key fact:

PROPOSITION 5.2. *Let (x,a) be a truncated Γ-system, pick any n such that $n+k$ is even. Then the product of matrices along the staircase path*

$$X(n,k)A(n,k)X(n+2,k)A(n+2,k)\dots X(n+2(k-1),k)A(n+2(k-2),k) \quad (14)$$

is anti-lower-triangular, i.e. it has the property that all of its $[j,l]$ entries with $j+l<2k$ vanish.

Remark. A similar statement holds for the Southern edge: the product

$$A(n,r+1)X(n,r+1)\dots X(n+2(k-1),r+1)A(n+2(k-2),r+1)$$

is anti-upper-triangular.

Proof. We give the proof for the case $k=3$, the general case being analogous. Without loss of generality we can choose $n=-1$. Then we start with

$$X(-1,3)A(-1,3)X(1,3)A(1,3)X(3,3)A(3,3)$$

and apply the (13) to each of the three products of the form XA. We end up with

$$A(-2,2)_{\oplus 1}\widehat{\Sigma}_{34}X(0,2)_{\oplus 1}A(0,2)_{\oplus 1}\widehat{\Sigma}_{34}X(2,2)_{\oplus 1}A(2,2)_{\oplus 1}\widehat{\Sigma}_{34}X(4,2)_{\oplus 1},$$

where somewhat loosely, we denoted all the matrices $\widehat{\Sigma}_{34}$ by the same symbol, even though only their shapes coincide. Observe that an anti-lower-triangular matrix will remain such after multiplication by A on the left or by X on the right, thus we can omit these matrices from the formula. Now we proceed to apply the equality (13) for $k=2$ to the two products of the form XA in this formula. We obtain

$$\widehat{\Sigma}_{34}A(-1,1)_{\oplus 2}\widehat{\Sigma}_{23}X(1,1)_{\oplus 2}\widehat{\Sigma}_{34}A(1,1)_{\oplus 2}\widehat{\Sigma}_{23}X(3,1)_{\oplus 2}\widehat{\Sigma}_{34}.$$

Observe that because of their direct sum structure the matrices $\widehat{\Sigma}_{34}$ commute with any matrix of the form $M_{\oplus 2}$. Finally, noting that for $k=1$ the equality (13) simply means that $X(1,1)_{\oplus 2}A(1,1)_{\oplus 2}=\widehat{\Sigma}_{12}$ we can rewrite (14) as

$$L\widehat{\Sigma}_{34}\widehat{\Sigma}_{23}\widehat{\Sigma}_{12}\widehat{\Sigma}_{34}\widehat{\Sigma}_{23}\widehat{\Sigma}_{34}U,$$

where L is a lower-, U is an upper-triangular matrix. This formula mimics the standard way of writing the longest element of the symmetric group as a product of nearby transpositions. We leave it as an exercise to check that such a matrix is anti-lower-triangular. □

This result has the following beautiful corollary. Because of the symmetry property of the system, we can assume without loss of generality that $r \geq k$. Recall that in our truncated system, the values of i will vary between 1 and $k+r+2$, however, those sites for which the matrices $X(n,i)$ and $A(n,i)$ are defined have $k \leq i \leq r+1$. We will call the edges of Γ for which the matrices are defined *regular*. Also, we will call *regular* those vertices of Γ which are endpoints of these edges.

Just as in the infinite case, we can define the parallel transport $\mathrm{PT}(p_1, p_2)$ between two regular vertices p_1, p_2, even when the Γ-system is truncated, by multiplying the $k+1$-by-$k+1$ matrices from one point to the other along any path consisting of regular edges. Again, this is well-defined because of the flatness condition.

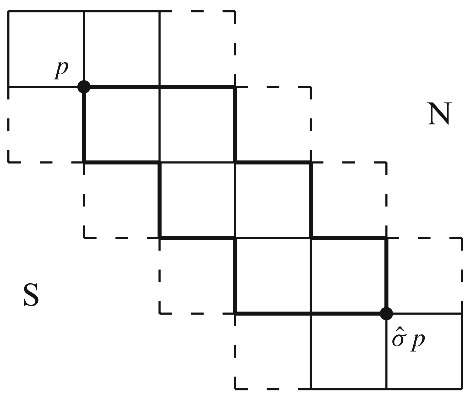
Now we can formulate an important corollary of Proposition 5.2.

COROLLARY 5.3. *Define a transformation of the ni-plane via the formula*

$$\hat{\sigma}(n,i) = (r+k+2+n, r+k+2-i).$$

Pick a regular vertex p of Γ; then the vertex $\hat{\sigma}p$ is also regular, and the parallel transport $\mathrm{PT}(p, \hat{\sigma}p)$ *is anti-diagonal.*

The following picture, showing the $k=2$, $r=3$ case, will be helpful. Here the regular part of Γ is shown with thin solid lines



Introduce the notation $\delta(p) = \mathrm{PT}(p, \hat{\sigma}p)$. Consider the two paths between p and $\hat{\sigma}p$ shown on the picture. Taking the northern path, we see that $\delta(p)$ has a representation

$$\delta(p) = A_1 \ldots A_l \bar{L} X_1 \ldots X_{l'},$$

where $\bar{L}$ is a matrix of the form (14), thus it is anti-lower-triangular. This implies that $\delta(p)$ is also anti-lower-triangular.

On the other hand, going along the southern path, we have

$$\delta(p) = Y_1 \dots Y_{l'} \bar{U} B_1 \dots B_l,$$

where $\bar{U}$ is anti-upper-triangular; this implies that $\delta(p)$ is also anti-upper-triangular.

The two statements together imply that $\delta(p)$ is anti-diagonal.

6. The Completion of the Proof

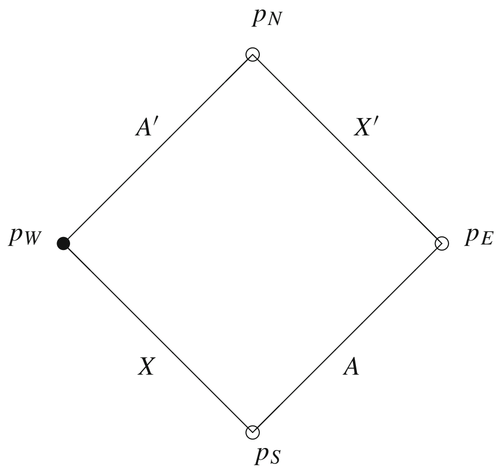
First we reformulate the periodicity property for the shifted system. Let us extend the transformation $\hat{\sigma}$ from the ni-plane to the whole lattice Λ via

$$\hat{\sigma}(n, i, j) = (r + k + 2 + n, r + k + 2 - i, k + 1 - j).$$

Then the periodicity property is equivalent to saying that the z-system is invariant under this transformation.

We will show this by proving an invariance property for the corresponding truncated Γ-system.

We again assume that $r \geq k$. Consider a square in the graph Γ consisting of regular edges. Introduce the following notation for the vertices and the edge matrices:



Thus we have

$$p_W \xrightarrow{A'} p_N \xrightarrow{X'} p_E \xleftarrow{A} p_S \xleftarrow{X} p_W.$$

Similarly, we set the notation

$$\hat{\sigma} p_W \xrightarrow{Y} \hat{\sigma} p_N \xrightarrow{B} \hat{\sigma} p_E \xleftarrow{B'} \hat{\sigma} p_S \xleftarrow{A'} \hat{\sigma} p_W.$$

We have the obvious equalities

$$\begin{aligned} A'\delta(p_N) &= \delta(p_W)Y, \\ A\delta(p_E) &= \delta(p_S)Y', \\ X\delta(p_S) &= \delta(p_W)B', \\ X'\delta(p_E) &= \delta(p_N)B. \end{aligned} \tag{15}$$

Denoting the diagonal part of a matrix M by ΔM, and its anti-diagonal part by $\bar{\Delta} M$, we see that

$$\begin{aligned} \Delta A'\bar{\Delta}\delta(p_N) &= \bar{\Delta}\delta(p_W), \\ \Delta A\bar{\Delta}\delta(p_E) &= \bar{\Delta}\delta(p_S), \\ \bar{\Delta}\delta(p_S) &= \bar{\Delta}\delta(p_W)\Delta B', \\ \bar{\Delta}\delta(p_E) &= \bar{\Delta}\delta(p_N)\Delta B. \end{aligned} \tag{16}$$

If w is the longest Weyl element, i.e. the anti-diagonal matrix with only 1s on the anti-diagonal, then for a diagonal matrix D, the matrix wDw is again diagonal, with the sequence of the diagonal elements reversed. The above equations then easily imply that

$$\Delta A \Delta A'^{-1} = w\Delta B' \Delta B^{-1} w.$$

As these are all diagonal matrices this equation simply means the sequence of equations

$$\frac{a_j}{a'_j} = \frac{b'_{k+2-j}}{b_{k+2-j}},$$

which is the periodicity property for the truncated Γ-system.

Finally, it easily follows from (11) that

$$\frac{1-z_-}{1-z} = \frac{a'}{a} \quad \text{if } 1 < j < k+1,$$

while according to (12), at the two extremes, we have

$$\frac{1}{1-z} = \frac{a'}{a} \quad \text{if } j=1, \quad \text{and} \quad 1-z_- = \frac{a'}{a} \quad \text{if } j=k+1.$$

This immediately implies the periodicity for the z variables lying over the regular strip. As long as $r \geq k$, these “regular” variables clearly determine the remaining values, and thus the proof is complete.

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